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Introduction to nonlinear dynamics

1) Introduction

Daily experience tells us that for many physical systems small changes in the initial conditions lead to small changes in the outcome. If we drive a car and turn the steering wheel a little, our course will differ only slightly from that which the car would have taken without the change. But there are cases for which the opposite is true: for a coin which is placed on its rim a slight touch is sufficient to determine the side on which it will fall. Thus the sequence of heads and tails which we obtain when tossing a coin, exhibits an irregular or chaotic behaviour in time, because extremely small changes in the initial conditions can lead to completely different outcomes. So we understand that a sensitive dependence on the initial conditions, results in a chaotic time-behaviour.

Dynamical systems are described by deterministic differential or difference equations, which give us a prescription for calculating their future behaviour from given initial conditions. Although these equations uniquely determine the time evolution of a state of the system from a knowledge of its previous history, it is possible for the dynamical system to exhibit irregular or chaotic motion. This behavior can be seen if the equations of the system are nonlinear, meaning that the output of an operation is not proportional to the input. In the real world, most of the systems are described by nonlinear equations and the main goal is to predict the behaviour of a system after a long period of time. Thus we understand that chaotic behaviour is by no means exceptional but a typical property of many systems. Such behaviour has, for example, been found in periodically stimulated cardiac cells, in electronic circuits, at the onset of turbulence, in chemical reactions, in lasers, in accelerator physics, in celestial mechanics and in many biological, meteorological and economic models.

In this paper we will try to describe the basic elements of nonlinear dynamical systems, taking a first look in the "magic" world of chaotic behaviour. In section 2 we define the basic concepts of dynamical systems like: phase space, Poincare surface of section, Poincare map, and we refer to the two classes of dynamical systems the conservative and the dissipative ones. We also give a rather qualitative definition of chaos.

In section 3 we deal with simple conservative systems of low dimentionality like the pendulum and the Henon-Heils system. We refer to the stability of periodic orbits in these systems and describe in detail a route that leads to large scaled chaos, namely the period doubling (or Feigenbaum) sequence.

Section 4 is devoted to dissipative systems. We give many simple examples in order to clarify the various types of attractors that these systems exhibit and we study in more detail the most interesting type of them i.e. the strange attractor.

2) Dynamical systems

A *dynamical system* is any physical system whose state is defined at any time by the values of N variables $x_1, x_2, \ldots, x_N$, and its evolution in time is determined by a set of differential equations:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_N, t) \quad , \quad i = 1,2,\ldots, N \qquad (1).$$

The N variables $x_1, x_2, \ldots, x_N$, can be any physical quantities like position, velocity, angle, temperature, pressure etc. If the system (1) can be solved for given initial conditions, the variables x_i are expressed as functions of time t. This means that the states of the system are determined by a given initial state. But, for the majority of the physical systems the corresponding equations (1) are in general nonlinear and can not be solved explicitly (for every value of time and any initial conditions of the variables) using the known methods of mathematical analysis.

The N-dimensional space with coordinates these N variables is called *phase space* of the dynamical system. A point in the phase space represents a certain state of the dynamical system, thus the evolution of the system in time is represented by a curve in the phase space

The dynamical systems are divided in two classes: a) the *conservative* and b) the *dissipative* systems. In the first systems any phase space volume is preserved as time varies (Liouville theorem), which means that although it is moved in the phase space, as time evolves, its shape can change but the volume remains the same. On the other hand in dissipative systems any phase space volume changes as time increases.

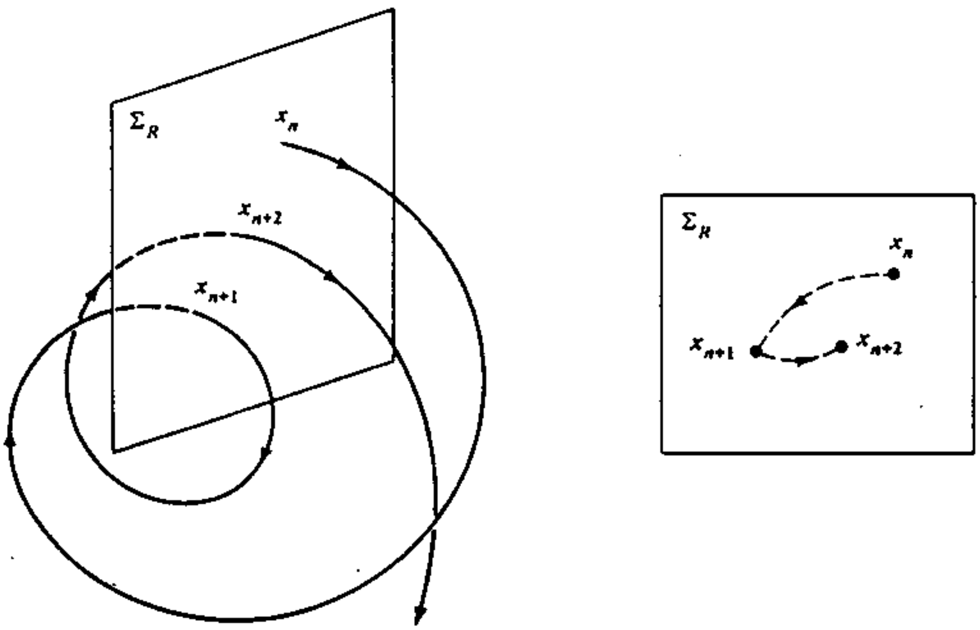


Figure 1: Intersections of a trajectory with the Poincare surface of section (P.s.s) Σ_R.

The study of the behaviour of dynamical systems both conservative and dissipative, is generally a complicated problem and for that reason several methods have been developed. One of the most used method is the construction of the *Poincare surface of section* (P.s.s). This method is mainly well working in systems with few degrees of freedom. In order to understand the P.s.s. method let us consider a conservative dynamical system of 2 degrees of freedom. This system is described by four differential equations of the form (1). We also consider that the system is autonomous, which means that the functions f_i , $i = 1,2,3,4$ in equation (1) do not depend explicitly on time t. We notice that we can consider any conservative system as autonomous without loss of the generality. We choose a 2-dimensional surface Σ_R (Fig. 1) in the 4-dimensional phase space and label its two sides (say left and right). Then we can limit the study of the dynamical system in the study of the successive intersections of the trajectories with this surface. The intersections are generated each time the trajectory cross the surface in a particular sense (say from left to right). The surface Σ_R is called Poincare surface of section. Considering every point of the P.s.s. as an initial point (T_0) of a trajectory of the dynamical system we can find its successive intersections with the P.s.s. ($T_1, T_2, ...$). Thus we define a map of the surface of section on itself, by corresponding every point (T_0) of the P.s.s. to the first intersection (T_1) with the surface, of the trajectory with initial condition T_0. This is the so called *Poincare map.*

We stress out, that in general, we can not find the difference equations of the Poincare map for a given dynamical system. The Poincare map is found only by solving numerically the differential equations of the system, finding the successive intersections of various trajectories with the surface of section.

Many times instead of studying systems of differential equations, we work with *maps* or in other words a set of difference equations (discrete time):

$$x_{i,t+1} = f_i(x_{1,t}, x_{2,t}, \ldots, x_{N,t}) \quad , \quad i = 1,2,\ldots,N \quad , \; t = 0,1,2,\ldots \qquad (2).$$

These equations can be considered as the Poincare map of some dynamical system. So a 2-dimensional conservative map corresponds to a dynamical system which is defined by four differential equations. Maps exhibit similar behaviour to dynamical systems and in some cases are preferred because their study is easier and less time consuming in a computer.

The most important feature of nonlinear dynamical systems is the chaotic behaviour that they exhibit. In order to give a qualitative definition of *chaos* we can say that by this term, we denote the disordered and irregular motion that can be seen in these systems. We remark that linear differential or difference equations do not lead to chaos and that nonlinearity is a necessary but not sufficient condition for the generation of chaotic motion. The most common source of irregularity is the property of the nonlinear systems of separating initially close trajectories, exponentially fast in a bounded region of phase space. This property is also called *sensitive dependence on the initial conditions,* but we make clear that today it is not the only condition for a motion to be called chaotic.

3) Conservative systems

Conservative systems are very important since they describe, by differential equations or maps, the motion of bodies in our solar system, the motion of satellites around earth or the motion of stars in galaxies, the motion of particles in accelerators or the behaviour of particles in plasma machines, etc.

A very interesting issue is the long-time behaviour of conservative systems, because for instance, by knowing this behaviour we can answer the question whether the solar system and galaxies are stable under small perturbations or whether they will eventually collapse or disperse to infinity. The long-time limit involved in this problem is of the order of the age of the universe. But "long" times are much shorter in accelerator physics, where the particles make many revolutions in fractions of a second. In such systems irregular or chaotic motion is to be avoided, which is possible if the long-time behaviour of these conservative systems is known.

At first we consider a simple physical problem of 1 degree of freedom: the motion of a pendulum of length l and mass m (Fig. 2). The phase space of this system is 2-dimensional, because the variables needed to define its state are the angle ϑ and the corresponding momentum:

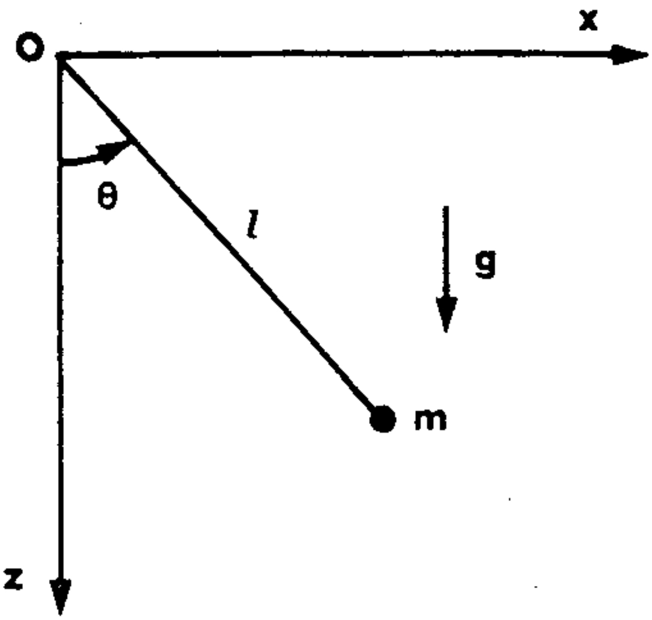
$$p = ml^2\dot{\vartheta} \qquad (3).$$

The energy h of the system

$$h = \frac{p^2}{2ml^2} - mgl \cdot \cos(\vartheta) \tag{4}$$

is constant. The equations of the dynamical system are:

$$\dot{\vartheta} = \frac{p}{ml^2} \quad , \quad \dot{p} = -mgl \cdot \sin(\vartheta) \tag{5}.$$



<u>Figure 2:</u> **A simple physical system: the pendulum.**

The phase space diagram of the system can be made using equation (4) from which we get:

$$p = \pm\sqrt{2ml^2(h + mgl \cdot \cos\vartheta)} \tag{6}.$$

We can see three different types of motion:

a) If $h < mgl$ the motion is bounded in ϑ (libration) between angles $\pm\vartheta_0$ where:

$$\vartheta_0 = \arctan(-\frac{h}{mgl}) < \pi \tag{7}.$$

So this motion corresponds to the closed curves l in figure 3.

b) If $h > mgl$ the motion is unbounded in ϑ (rotation) corresponding to the curve r in Fig. 3.

c) For $h = mgl$ we get the separatrix motion (curve s in Fig. 3) in which the oscillation period becomes infinite. Equation (6) gives:

$$p = \pm\sqrt{2m^2gl^3(1 + \cos\vartheta)}.$$

Using equation (3) and assuming that for $t = 0$ the angle was ϑ_0 we get :

$$\vartheta = 4 \cdot \arctan[\tan(\frac{\vartheta_0 + \pi}{4}) \cdot e^{\pm\sqrt{\frac{g}{l}}t}] - \pi \tag{8}.$$

So if $p(0) > 0$ then the orbit tends to point A for $t \to +\infty$ and to point B for $t \to -\infty$. This motion corresponds to the upper curve s in Fig.3. If $p(0) < 0$, the orbit tends to point B for $t \to +\infty$ and to point A for $t \to -\infty$ (lower curve s in Fig. 3).

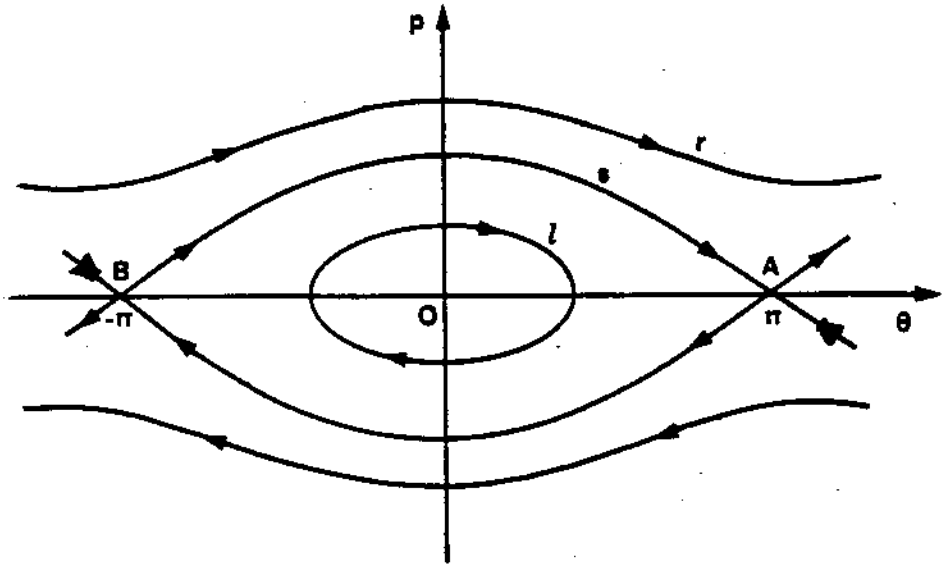


Figure 3: The phase space diagram of the pendulum.

If the state of the system is represented by points O, A or B then this state does not change. Point O is a *stable* or *elliptic* singular point. A phase space trajectory near an elliptic singular point remains in its neighborhood, while a trajectory near a *unstable* or *hyperbolic* point (point A and B in Fig.3) diverges from it.

We emphasize that the motion of any conservative system with 1 degree of freedom (2-dimensional phase space) is never chaotic, as we saw in the phase space of the pendulum where the motion is always ordered. In this case the system can be solved in principle.

Conservative systems with more than 1 degree of freedom exhibit in general regions in the phase space where the motion is irregular or chaotic although there also exist regions where the motion is ordered.

For conservative systems with 2 degrees of freedom the phase space is 4-dimensional, which means that we can not have inspection of it. Thus we study the behaviour of the system in the 2-dimensional surface of section (Fig. 4). These systems exhibit similar behaviour to 2-dimensional conservative maps. As we understand from Fig. 4 a periodic orbit of period n (which means that the orbit makes n cycles before passing from its initial point) is represented on the P.s.s. by n points, while the irregular orbit produce many scattered points on the surface of section.

One of the first models of classical mechanics which was found to exhibit regions of irregular motion was a system of two nonlinear coupled harmonic oscillators with energy E:

$$E = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3 \qquad (9),$$

which was studied by Henon and Heils (1964). In figure 5 we see the surface of section of this system for some values of the energy E (which acts as a parameter of the system). For small values of the energy the motion seems to be ordered in all the phase space (although there are very small regions of chaotic motion which can not be seen at the present scale). As the energy increases the regions of chaotic motion increase too. Thus in the phase space we can see islands of ordered motion near stable periodic orbits and chaotic regions of irregularly distributed points near unstable periodic orbits. The orbits near a stable periodic orbit behave in a similar way to the periodic one and remain to its neighborhood, forming closed curves on the P.s.s. (*invariant curves*) around the periodic fixed points. On the other hand orbits near unstable periodic orbits diverge from it.

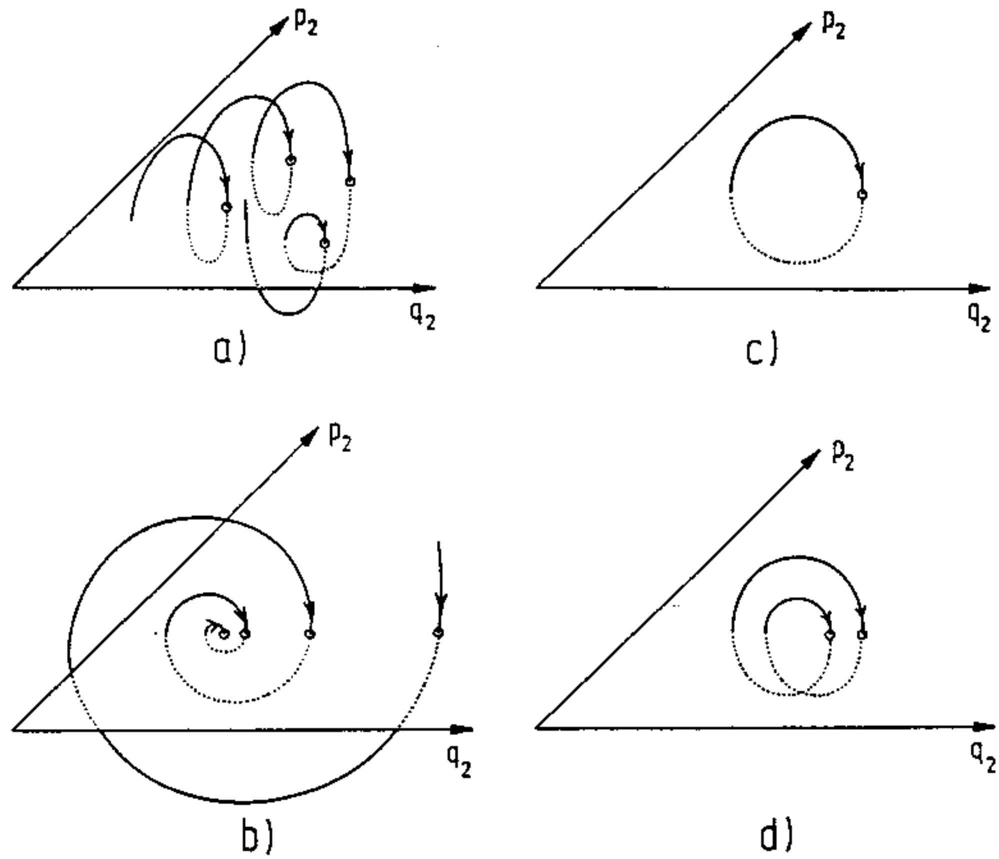


Figure 4: Qualitatively different trajectories can be distinguished by their Poincare sections: a) chaotic motion b) approach of a fixed point c) periodic orbit and d) periodic orbit of period two.

A very important route to large scaled chaos is the *Feigenbaum sequence of period doubling bifurcations*. As a parameter (let as call it r) of a dynamical system changes a stable periodic orbit of period n can become unstable (for $r = r_1$), giving birth to a stable periodic orbit of period $2n$. For a little larger value of the parameter ($r = r_2$) this periodic orbit becomes unstable an a new periodic orbit of period $4n$ is born (Fig. 6). This period doubling sequence continues up to a certain value of the parameter ($r = r_\infty$) over which all the periodic orbits of this sequence are unstable leading to large chaotic regions. This is a very general procedure which occurs both in conservative and dissipative systems. The so called *Feigenbaum ratio*:

$$\delta \equiv \lim_{k\to\infty} \frac{r_{k-2} - r_{k-1}}{r_{k-1} - r_k} , \quad k \geq 3 \qquad (10),$$

tends to universal constants which are 8.72 for conservative systems and 4.67 for the dissipative ones. Also the ratio $\frac{d_{k-1}}{d_k}$, $k \geq 2$ of distances d_k , $k = 1,2,3,\ldots$, between two symmetric points of the orbits (Fig. 6), tends to universal constants also: 4.018 for conservative systems and 2.503 for dissipative systems.

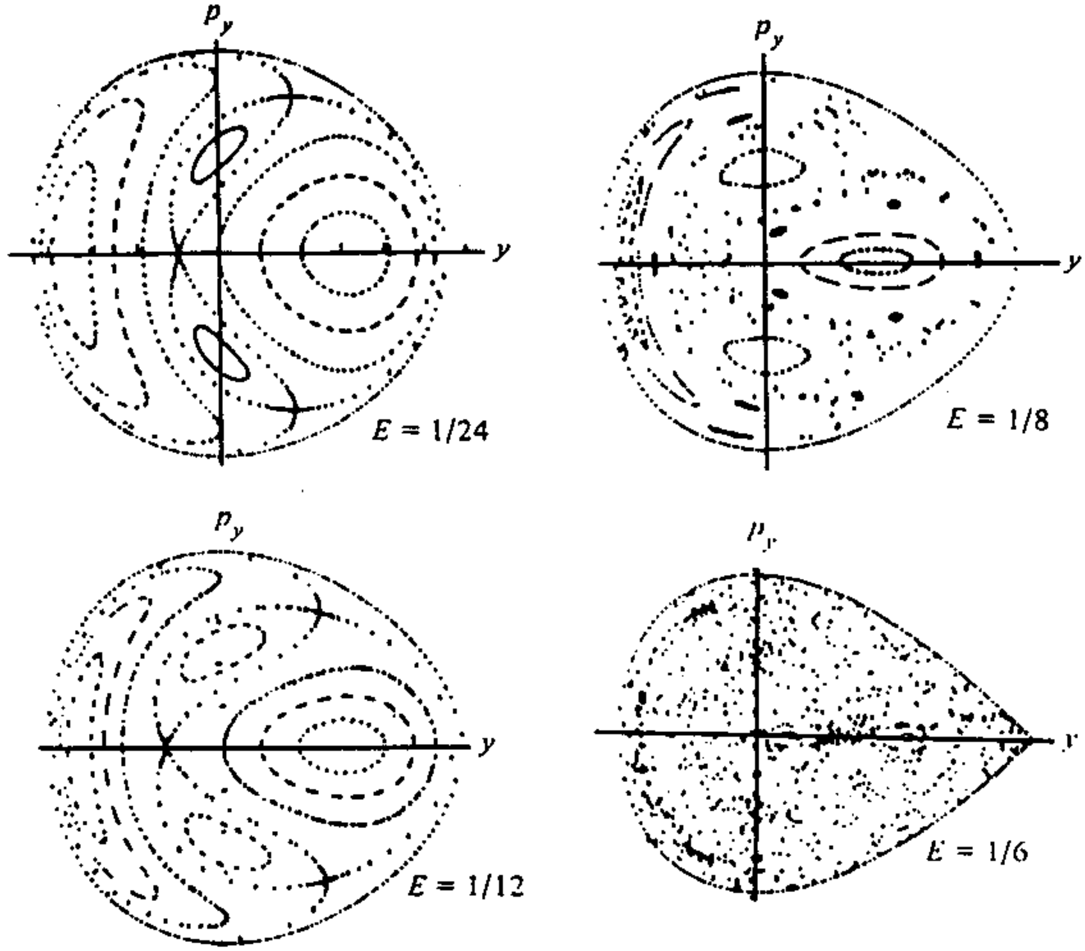


Figure 5: Poincare maps for the Henon-Heils system for several values of energy E.

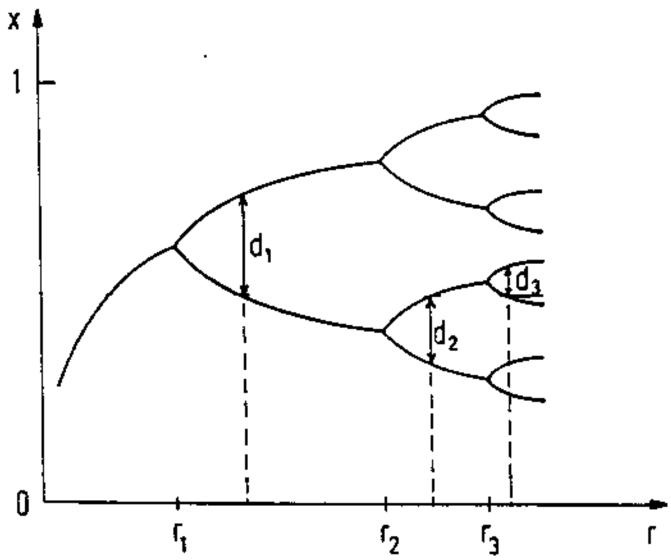


Figure 6: A qualitative scheme of a period doubling bifurcation sequence, where r is the parameter of the system and x a dynamical variable.

4) Dissipative systems

A dissipative system is characterized by continued contraction of the phase space volume with increasing time. The rate of change of volume locally may be positive (expanding) or negative (contracting) but averaged over an orbit the volume contracts. This leads to contraction onto a surface of lower dimensionality than the original phase space, which is called *attractor.* Further more as a system parameter changes the motion on the attractor may change from regular to chaotic.

Following Lanford (1981) we call a subset X of the phase space an attractor if:
a) X is invariant under the evolution of the dynamical system,
b) there is an open neighborhood around X (*basin of attraction)* that shrinks down to X as time increases, and
c) X can not be decomposed into nonoverlapping invariant pieces.
The basin of attraction of X is the set of states in phase space that approach X as $t \to \infty$. If there exist many attractors in the phase space, all initial states (except for a set of measure zero) lie in the basin of one of them (Fig. 7).

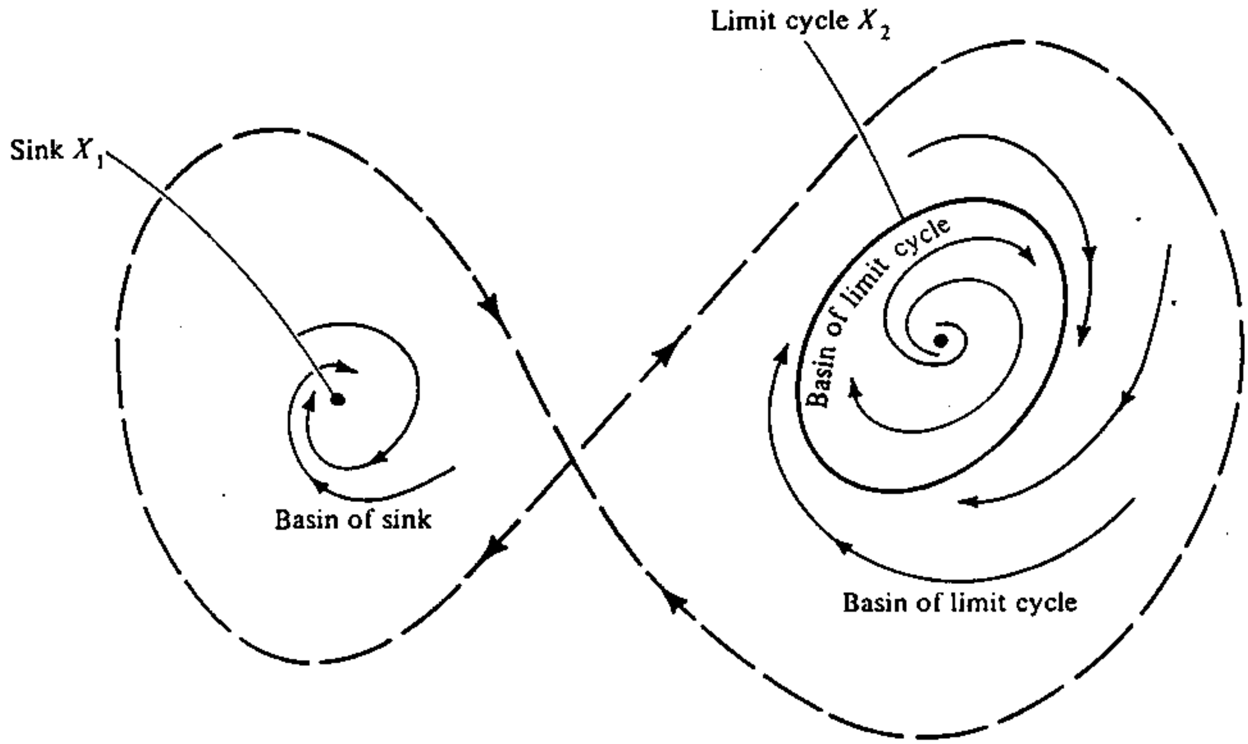


Figure 7: Attractors in phase space and their basins of attraction. A sink and a limit cycle are shown.

For 1-dimensional dissipative systems the only possible attractors are stable fixed points or sinks (point X_1 in Fig. 7). The fixed points of the dynamical system:

$$\frac{dx}{dt} = V(x, \mu) \qquad (11),$$

where μ is a parameter has fixed points at $\frac{dx}{dt} = 0 \Rightarrow V(x, \mu) = 0$. For example for

$V(x,\mu) = \mu - x^2$, there exist a stable fixed point $x = +\sqrt{\mu}$ and an unstable one $x = -\sqrt{\mu}$ for $\mu > 0$. For $\mu < 0$ there are no fixed points of real x. The creation of these two fixed points as μ passes through zero is illustrated in figure 8a and in an example of a tangent bifurcation. There are several other types of bifurcations. For $V(x,\mu) = \mu x - x^2$ there is an exchange of stability (Fig. 8b), with the sink (solid line) passing from the fixed point at $x = 0$ to the fixed point at $x = \mu$ as μ passes through zero from left to right. For $V(x,\mu) = \mu x - x^3$ there is a pitchfork bifurcation (Fig. 8c). The sink at $x = 0$ is destroyed and two new sinks at $x = \pm\sqrt{\mu}$ are created. In addition there can be a reverse pitchfork bifurcation (Fig. 8d). Except for the nongeneric cases, these are the only bifurcations present in 1-dimensional dissipative systems.

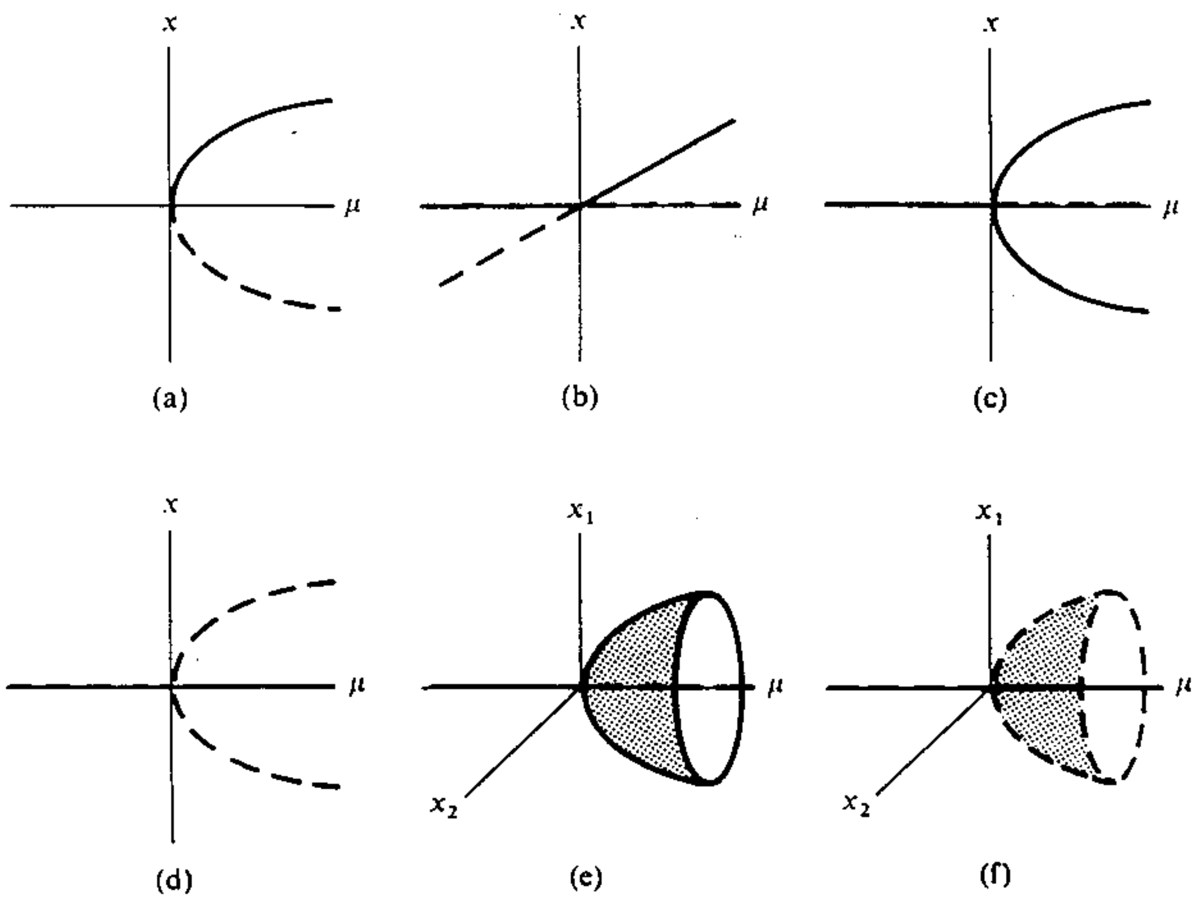


Figure 8: Bifurcations in 1- and 2-dimensional systems. a) Tangent, b) exchange of stability, c) pitchfork, d) anti-pitchfork, e) Hopf and f) inverted Hopf. Cases a)-d) are generic for 1-dimensional systems; cases a)-f) are generic for 2-dimensional systems. A solid (dashed) line denotes a stable (unstable) orbit.

For 2-dimensional dissipative system with bounded phase space there are only two kinds of attractors: a) fixed points or sinks , and b) periodic solutions (simple closed curves) or *limit cycles* (curve X_2 in Fig. 7). The formation of a limit cycle can be studied for example in the system (in polar coordinates):

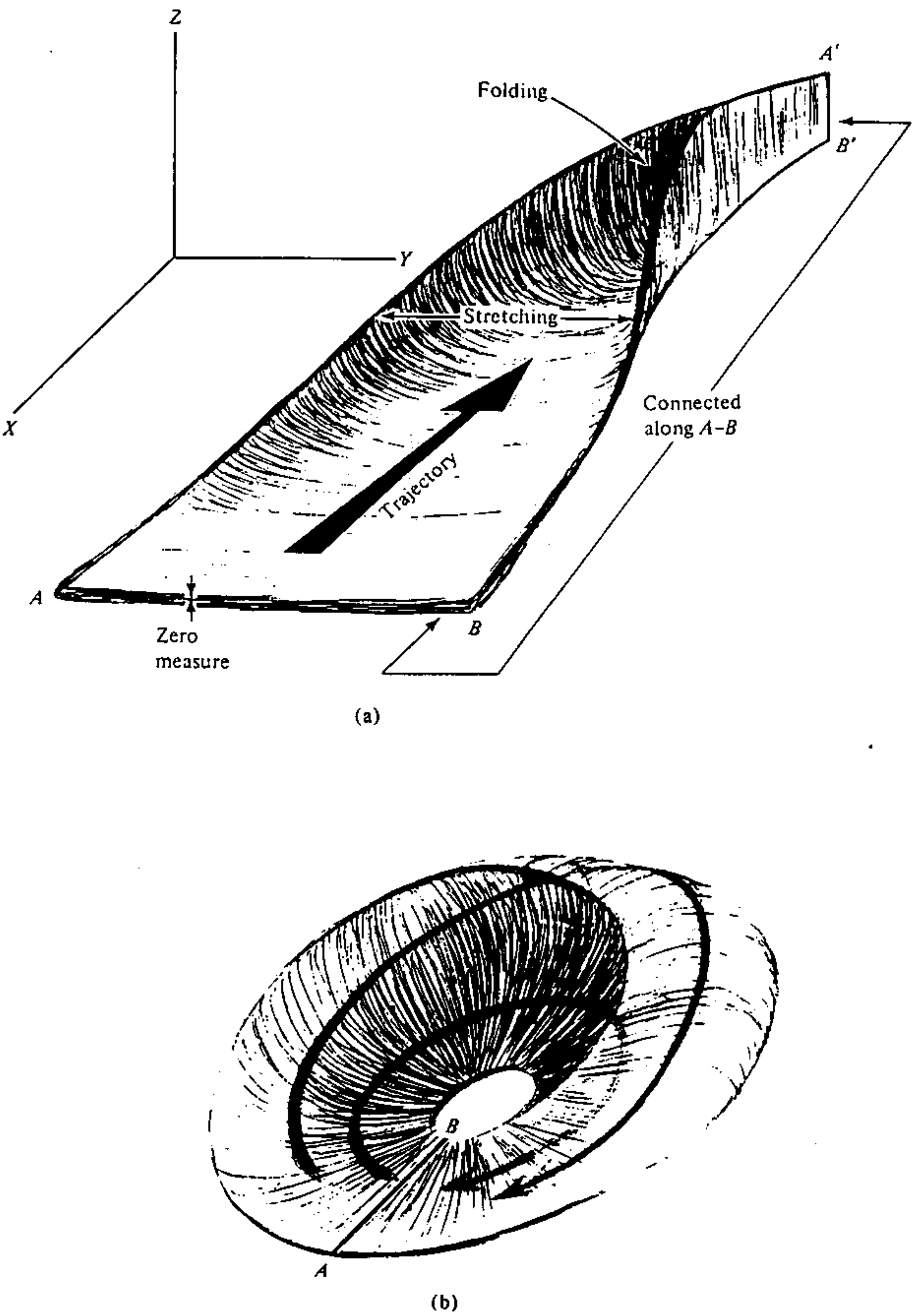


Figure 9: Qualitative illustration of a strange attractor. a) An infinite layer of ribbons is stretched and folded. b) The structure is joined head-to-tail and embedded in 3-dimensional space.

$$\frac{dr}{dt} = \mu r - r^2 \qquad (12a)$$

$$\frac{d\vartheta}{dt} = \omega_0 > 0 \qquad (12b).$$

For $\mu < 0$ the right-hand of (12a) is always negative and the motion spirals into the sink at $r = 0$. For $\mu > 0$ the right-hand side is positive near $r = 0$ and the fixed point is no longer attractor. In this case we note that the right-hand side is positive for $r < \mu$, leading to an increase in r with time and negative for $r > \mu$, leading to an decrease in r with time. Thus the motion is attracted to the limit cycle $r(t) = \mu$, $\vartheta(t) = \omega_0 t + \vartheta_0$, yielding stable motion on a circle of radius μ. The change from sink to limit cycle as μ passes through zero is called a Hopf bifurcation, (Fig. 8e). The inverted Hopf bifurcation (Fig. 8f) is also possible. The only bifurcations that are present generically in 2-dimensional dissipative systems are shown in Fig. 8.

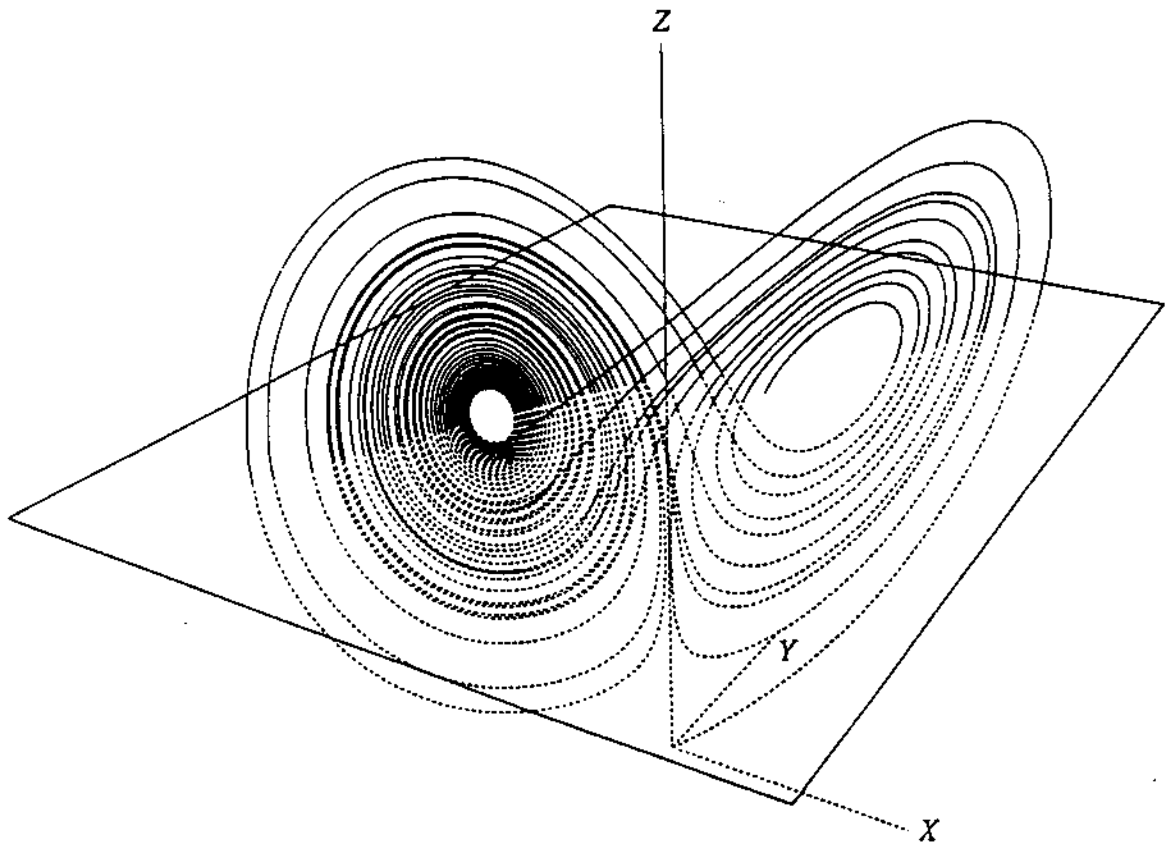
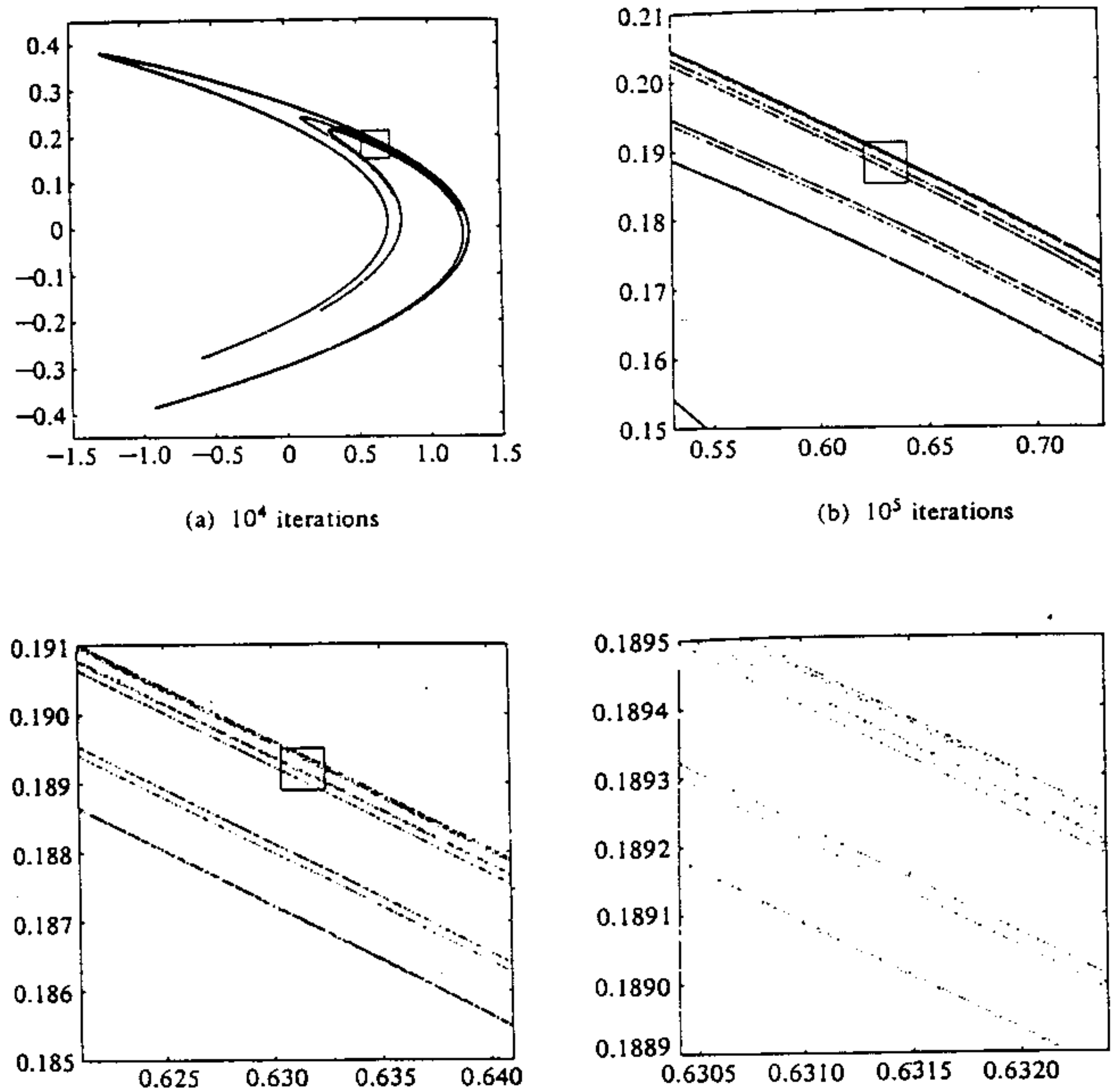


Figure 10: The Lorenz attractor. The horizontal plane is at $Z = 27$.

For dissipative systems with dimension $N \geq 3$ in addition to sinks and limit cycles can also exist attractors that have very complicated geometric structures. These structures can be characterized as having a fractional dimension and are usually called *strange attractors*. The motion on strange attractors is chaotic. The property which makes the attractor strange is the sensitive dependence on the initial conditions: i.e. despite the contraction in volume, lengths need not shrink in all directions, and points, which are arbitrarily close initially, become exponentially separated at the attractor for sufficiently long times.

To visualize a strange attractor we imagine a 3-dimensional structure in the form of a layer containing infinitely many 2-dimensional sheets. The layer expands along its width and folds over on itself, as shown in figure 9a. The two ends (A B and A' B' in the figure)are smoothly joined together. Since A' B' having two distinct sheets, joins to A B , having one sheet, there must be infinitely many sheets for the joining to be smooth. The infinitely leaved structure, when smoothly joined and embedded in a 3-dimensional phase space, is shown in figure 9b. From the construction, it is seen that orbits are bounded despite the fact that nearby orbits diverge exponentially. Furthermore, the structure of the leaves is such that on finer and finer scales the basic leaf pattern reappears.



(a) 10^4 iterations

(b) 10^5 iterations

(c) 10^6 iterations

(d) 5×10^6 iterations

Figure 11; a)The Henon attractor, b) enlargement of the small square in (a), c) enlargement of the small square in (b) and d) enlargement of the small square in (c).

When a parameter changes in a dissipative system, the system may change from periodic motion to the chaotic motion on a strange attractor. In many cases this change proceeds by successive doublings of the period of the singly periodic motion to some limit, beyond which the attractor changes and becomes chaotic. Further change in the parameter can lead to an inverse process or the appearance of simple attractor basins with other symmetries.

An example of a strange attractor is given by the Lorenz model (1963):

$$\begin{aligned} \dot{X} &= -\sigma X + \sigma Y \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ \end{aligned} \qquad (13),$$

which is a simplified model which tries to describe the motion of a liquid when we heat it. The volume element contracts exponentially in time:

$$V(t) = V(0) \cdot e^{-(\sigma+b+1)t} \qquad (14).$$

A trajectory generated by the equations of the Lorenz model for $r = 28$, $\sigma = 10$ and $b = \frac{8}{3}$ is attracted to a boundary region in phase space (Fig. 10). The motion is very erratic i.e. the trajectory makes one loop to the right, then a few loops to the left, then again to the right, and so on. Also there exist a sensitive dependence of the trajectory on the initial conditions i.e. a nearby orbit makes a completely different number of loops to the two regions of the attractor.

Another example of a strange attractor is due to Henon (1976) who considered the simple case of the map:

$$\begin{aligned} x_{n+1} &= y_n + 1 - \alpha x_n^2 \\ y_{n+1} &= b x_n \end{aligned} \qquad (15).$$

This map can be considered as the P.s.s. of a 3-dimensional phase space. For one iteration the area contracts by a factor b.

For $b = 0.3$ and $\alpha = 1.4$ the map has the strange attractor seen in figure 11a, which is obtained for 10^4 iterations of the map. When the area within the small square in Fig. 11a is magnified, then 10^5 iterations yield the more detailed view of the attractor shown in Fig. 11b. When the square in Fig. 11b is magnified, 10^6 iterations yield still more detail of the structure across the leaves as shown in Fig. 11c. A final magnification in Fig. 11d for $5 \cdot 10^6$ iterations, shows again the scale invariance of this attracting structure: Figs. 11b-d, obtained by repeated magnifications, all look identical.

Of course there exist many other dissipative systems that exhibit strange attractors, but the two examples mentioned are simple enough and have been studied widely.

General references

Helleman, R.H. and Ioos, G. (eds.) (1983): Les Houches Summerschool on " Chaotic behaviour in deterministic systems ". North-Holland. Amsterdam.

Lichtenberg, A.J. and Lieberman, M.A. (1982): " Regular and stochastic motion ". Springer-Verlag. New York.

Shuster, H.G. (1989): " Deterministic chaos. An introduction ". VCH Verlagsgesellschaft. Weinheim, Germany.

References

Feigenbaum, M.J. (1978). *J. Stat. Phys.* **19**, 25.

Feigenbaum, M.J. (1979). *J. Stat. Phys.* **21**, 669.

Henon, M. and Heils, C. (1964). *Astron. J.* **69**, 73.

Henon, M (1976). *Commun. Math. Phys.* **50**, 69.

Lanford, O.E. (1981) In " Hydrodynamic instabilities and the transition to turbulence ". Swinney, H.L. and Gollub, J.P. (eds.), Springer-Verlag, New York, Chapter 2.

Lorenz, E.N. (1963). *J. Atmos. Sci.* **20**, 130.

Poincare, H. (1892) : " Les methodes nouvelles de la mecanique celeste ". Gauthier-Villars. Paris.